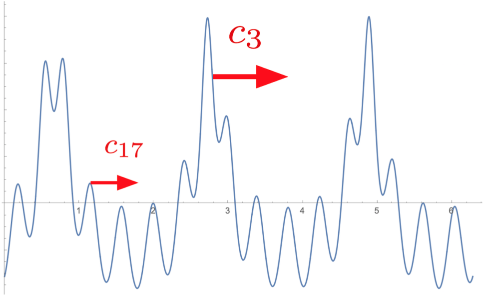
Quantization of Benjamin-Ono periodic traveling waves



Brown ⊕ BU ⊕ UMass Amherst
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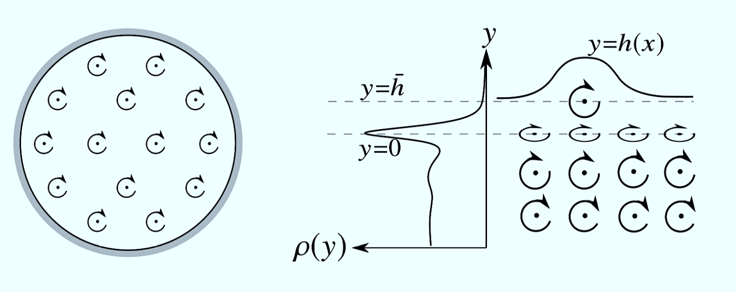
Classical Benjamin-Ono [BO] Equation: Motivation

The *classical Benjamin-Ono equation* for $v(x,t;\varepsilon_0)$: for coupling $\varepsilon_0 \in \mathbb{R}$,

$$\boxed{\partial_t v + v\partial_x v = -\tfrac{\varepsilon_0}{2} J[\partial_x^2 v]}$$

spatial Hilbert transform $Je^{\mathbf{i}kx} = -\mathbf{i}\mathrm{sgn}(k)e^{\mathbf{i}kx}$. Survey: Saut (2019).



☐ Bogatskiy-Wiegmann "Edge Wave and Boundary Layer of Vortex Matter." *Physical Review Letters* 122 (2019).

⋆ Conjecture: free boundary of chiral quantum vortex fluid ≈ quantum BO ⋆

What is a "Quantum System"?

A **quantum system** is defined by a scale $\hbar > 0$ and 3 ingredients:

1. $\mathbb{C}$-Hilbert space $(F, \langle\cdot,\cdot\rangle)$ — *state space*
2. Self-adjoint operator $\widehat{H}$ on F — *quantum Hamiltonian*
3. Set $\mathbb{O}$ of self-adjoint operators $\widehat{O}$ on F — *quantum observables*

Spectral Theorem: [von Neumann 1932] Any self-adjoint $\widehat{O}$ on F and $\Psi \in F$ defines probability measure $\mu_{\Psi,\Psi}(\cdot|\widehat{O})$ on $\mathbb{R}$, the *spectral measure of* $\widehat{O}$ *at* Ψ*:*

$$\frac{\langle\Psi, e^{\mathbf{i}\tau\widehat{O}}\Psi\rangle}{\langle\Psi,\Psi\rangle} \overset{\star}{=} \int_{-\infty}^{+\infty} e^{\mathbf{i}\tau E} d\mu_{\Psi,\Psi}(E|\widehat{O}) \qquad \text{for all} \quad \tau \in \mathbb{R}.$$

Definition: Let $\widehat{O}\big|_{\Psi}$ denote the random variable with law $\mu_{\Psi,\Psi}(E|\widehat{O})$, characteristic function $\mathbb{E}\left[e^{\mathbf{i}\tau\widehat{O}\big|_{\Psi}}\right]$ in $\overset{\star}{=}$, and moments $\mathbb{E}[(\widehat{O}\big|_{\Psi})^p] = \frac{\langle\Psi,\widehat{O}^p\Psi\rangle}{\langle\Psi,\Psi\rangle}$.

Proposal: [Born 1926] The outcome of **measuring** a quantum system in state $\Psi \in F$ with observable $\widehat{O}$ is **random**: it is the random variable $\widehat{O}\big|_{\Psi}$!

$\star$ Random variables $\widehat{O}\big|_{\Psi}$ from *deterministic* $\widehat{O}$ *and* Ψ (no path integral) $\star$

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Dynamics: state $\Psi(t) \in F$ of quantum system **evolves in two ways**:

1. **In isolation**, $\Psi(t)$ solves the linear *Schrödinger equation*

$$\mathbf{i}\hbar\partial_t \Psi = \widehat{H}\Psi$$

which is both reversible and deterministic.

2. **Observed by** $\widehat{O}$, $\Psi(t)$ collapses instantaneously by *Born rule*

$$\Psi(t) \longrightarrow \text{project } \Psi(t) \text{ to random } \widehat{O}\Big|_{\Psi(t)} \text{ eigenspace of } \widehat{O}$$

which is both irreversible and stochastic.

⋆ Parallel: emergence of irreversibility in classical dynamical systems ⋆

Classical BO Equation on $\mathbb{T}$: Classical Hamiltonian

The *classical Benjamin-Ono equation* for $v(x,t;\varepsilon_0)$: for coupling $\varepsilon_0 \in \mathbb{R}$,

$$\partial_t v + v\partial_x v = -\frac{\varepsilon_0}{2} J[\partial_x^2 v]$$

spatial Hilbert transform $Je^{\mathbf{i}kx} = -\mathbf{i}\mathrm{sgn}(k)e^{\mathbf{i}kx}$. Survey: Saut (2019).

Problem: Study solutions $v(x,t;\varepsilon_0)$ which are 2π-periodic in x:

$$v(x,t;\varepsilon_0) = \sum_{k=-\infty}^{+\infty} V_k(t;\varepsilon_0)e^{-\mathbf{i}kx}.$$

Claim: Formally, a **classical energy (Hamiltonian) is conserved**: for

$$\boxed{H(\varepsilon_0)\Big|_v = \frac{1}{2}\sum_{j_1,j_2=1}^{\infty} V_{j_1}V_{j_2-j_1}V_{-j_2} + \frac{1}{2}\sum_{j=1}^{\infty}(\varepsilon_0 j - V_0)V_jV_{-j} + \frac{1}{6}V_0^3}$$

any solution of BO with dispersion coefficient ε_0 satisfies

$$\frac{d}{dt}H(\varepsilon_0)\Big|_{v(x,t;\varepsilon_0)} = 0.$$

⋆ Omit: the *symplectic space* is the *critical Sobolev space* with $s_c = -1/2$. ⋆

Quantum BO Equation: Hilbert Space of States

Observe: For real-valued solutions $v(x,t;\varepsilon_0)$ of classical BO, always have

$$V_{-k}(t;\varepsilon_0) = \overline{V_k(t;\varepsilon_0)} \qquad \text{and} \qquad \frac{d}{dt}V_0(t;\varepsilon_0) = 0$$

so can restrict attention to $V_k(t;\varepsilon_0) \in \mathbb{C}$ with $k = 1, 2, 3, \ldots$.

Vector Space: The ring R of polynomials in $V_1, V_2, V_3, \ldots$ has explicit basis

$$R = \mathbb{C}[V_1, V_2, V_3, \ldots] = \operatorname{span}\left\{ V_1^{d_1} V_2^{d_2} V_3^{d_3} \cdots \ : \ d_k \in \mathbb{Z}_{\geq 0} \text{ almost all zero} \right\}$$

Inner Product: Define $\langle \cdot, \cdot \rangle_\hbar$ on R by declaring this basis orthogonal and

$$\left|\left| V_1^{d_1} V_2^{d_2} V_3^{d_3} \cdots \right|\right|_\hbar^2 = \prod_{k=1}^{\infty} (\hbar k)^{d_k} d_k!$$

Choice #1: As *state space* $(F, \langle \cdot, \cdot \rangle_\hbar)$ for quantum BO on $\mathbb{T}$, choose to use F_R, the Hilbert space completion of R by $\langle \cdot, \cdot \rangle_\hbar$.

⋆ Omit: F_R defined by *standard Gaussian* on symplectic space $s_c = -1/2$. ⋆

Quantum BO Equation: Quantum Hamiltonian

Recall: Formally, a *classical energy (Hamiltonian)* is conserved:

$$H(\varepsilon_0)\Big|_v = \frac{1}{2}\sum_{j_1,j_2=1}^{\infty} V_{j_1}V_{j_2-j_1}V_{-j_2} + \frac{1}{2}\sum_{j=1}^{\infty}(\varepsilon_0 j - V_0)V_jV_{-j} + \frac{1}{6}V_0^3$$

Quantization: Unbounded operators on F_R = closure of $\mathbb{C}[V_1, V_2, V_3, \ldots]$:

$$\widehat{V}_k = \text{multiplication by } V_k \qquad k = 1, 2, 3, \ldots$$

$$\widehat{V}_{-k} = \hbar k \frac{\partial}{\partial V_k} \qquad k = 1, 2, 3, \ldots .$$

Claim: For fixed $V_0 \in \mathbb{R}$, let $\widehat{V}_0$ = multiplication by V_0. For any $\overline{\varepsilon} \in \mathbb{R}$,

$$\boxed{\widehat{H}(\overline{\varepsilon};\hbar) = \frac{1}{2}\sum_{j_1,j_2=1}^{\infty} \widehat{V}_{j_1}\widehat{V}_{j_2-j_1}\widehat{V}_{-j_2} + \frac{1}{2}\sum_{j=1}^{\infty}(\overline{\varepsilon} j - \widehat{V}_0)\widehat{V}_j\widehat{V}_{-j} + \frac{1}{6}\widehat{V}_0^3}$$

is a well-defined self-adjoint operator in F_R with discrete spectrum.

Choice #2: As *quantum Hamiltonian* $\widehat{H}$ for quantum BO on $\mathbb{T}$, choose to use the self-adjoint operator $\widehat{H} = \widehat{H}(\overline{\varepsilon}, \hbar)$ in $F = F_R$.

⋆ Use new variable $\overline{\varepsilon}$: anticipate *renormalization* of classical coupling ε_0 ⋆

Quantum BO Equation: Stationary Dynamics in Isolation

Recall: **In isolation**, the quantum BO equation for $\Psi(t;\overline{\varepsilon},\hbar)$ in F_R is

$$\mathbf{i}\hbar\partial_t\Psi = \widehat{H}(\overline{\varepsilon},\hbar)\Psi.$$

Eigenfunctions of $\widehat{H}(\overline{\varepsilon},\hbar)$ define **stationary solutions** of quantum BO.

Theorem: [Stanley 1989] Fix $\overline{\varepsilon} \in \mathbb{R}$ and $\hbar > 0$. Then

1. The eigenfunctions of the quantum BO Hamiltonian

$$\widehat{H}(\overline{\varepsilon};\hbar) = \frac{1}{2}\sum_{j_1,j_2=1}^{\infty} \widehat{V}_{j_1}\widehat{V}_{j_2-j_1}\widehat{V}_{-j_2} + \frac{1}{2}\sum_{j=1}^{\infty}(\overline{\varepsilon}j - \widehat{V}_0)\widehat{V}_j\widehat{V}_{-j} + \frac{1}{6}\widehat{V}_0^3$$

coincide with the *Jack polynomials* $P_\lambda(V_1, V_2, V_3, \ldots; \overline{\varepsilon}, \hbar)$ of Jack (1970).

2. These eigenfunctions are indexed by **partitions** λ, i.e. sequences

$$\lambda = (0 \le \cdots \le \lambda_2 \le \lambda_1) \quad \text{so} \quad \lambda_j \in \mathbb{Z}_{\geq 0} \quad \text{and almost all zero.}$$

3. The eigenvalues $E_\lambda(\overline{\varepsilon},\hbar)$ are explicit functions of the *Young diagram* of λ.

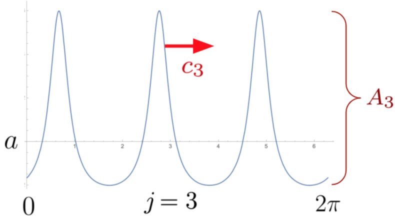
⋆ Main Result [M. 2019b]: relate to classical BO periodic traveling waves ⋆

Classical BO Periodic Traveling Waves: Formula

<u>Theorem</u> [Benjamin 1967, Ono 1975]: For any $a \in \mathbb{R}$, any $j = 1, 2, 3, \ldots$, $I_j \geq 0$, and $\chi_j \in \mathbb{R}$, the explicit periodic traveling waves 2π-periodic in x

$$\boxed{v^{(a, I_j, \chi_j)}(x, t; \varepsilon_0) = a + \varepsilon_0 j \left(1 - \frac{j}{j + 2I_j - 2\sqrt{I_j(j + I_j)}\cos(jx - \chi_j - \omega_j t)} \right)}$$

with phase velocity $c_j = \frac{\omega_j}{j} = a + \frac{\varepsilon_0}{2} j - \varepsilon_0 I_j$ **solve** $\partial_t v + \partial_x v = -\frac{\varepsilon_0}{2} J[\partial_x^2 v]$.



ε_0	dispersion coefficient
a	average height
j	wavenumber
$\varepsilon_0^2 I_j$	classical action*
A_j	amplitude $4\lvert\varepsilon_0\rvert\sqrt{I_j(j + I_j)}$
χ_j	initial phase shift

Prop. [M. 2019a] $(\varepsilon_0^2 I_j,\ \chi_j)$ *"action-angles" of* ***periodic*** *orbits* $v^{(I_j, \chi_j)}$.

⋆ In [M. 2019a], new approach to *interacting systems* of these waves ⋆

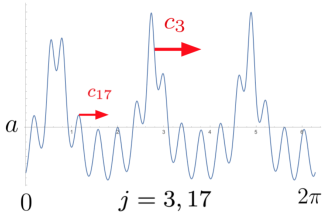
Classical BO Periodic Traveling Waves: Interactions

Theorem [Dobrokhotov-Krichever 1991]: For any $a \in \mathbb{R}$, $j = 1, 2, \ldots, N$, $\vec{I} = (I_j)_{j=1}^N$, $\vec{\chi} = (\chi_j)_{j=1}^N$, *there are explicit* $N \times N$ *matrices* $B^{\vec{I},\vec{\chi}}(t;\varepsilon_0)$ so

$$v^{(a,\vec{I},\vec{\chi})}(x,t;\varepsilon_0) = a - 2\varepsilon_0 \mathrm{Im}\partial_x \log \det_{N\times N}\left(\mathrm{Id} - e^{\mathbf{i}x}B^{\vec{I},\vec{\chi}}(t;\varepsilon_0)\right)$$

solve $\partial_t v + \partial_x v = -\frac{\varepsilon_0}{2}J[\partial_x^2 v]$. These solutions define *integrable systems* of *interacting* periodic traveling waves 2π-periodic in x with phase velocities

$$c_j = \frac{\omega_j}{j} = a + \frac{\varepsilon_0}{2}j - \varepsilon_0 I_j - \varepsilon_0 \sum_{p>j} I_p - \varepsilon_0 \sum_{p<j} \frac{p}{j} I_p.$$



ε_0	dispersion coefficient
a	average height
j	wavenumbers
$\varepsilon_0^2 I_j$	classical actions*
χ_j	initial phase shifts

Theorem [GK2019] $(\varepsilon_0^2 I_j, \chi_j)$ *"action-angles" of **quasi**-periodic* $v^{(\vec{I},\vec{\chi})}$.

⋆ In [M. 2019b], use [Gérard-Kappeler 2019] to reinterpret [Stanley 1989] ⋆

Classical Periodic Traveling Waves: Quantization

Recall: to model quantum effects, study quantum BO Hamiltonian on $\mathbb{T}$:

$$\widehat{H}(\overline{\varepsilon};\hbar) = \frac{1}{2}\sum_{j_1,j_2=1}^{\infty} \widehat{V}_{j_1}\widehat{V}_{j_2-j_1}\widehat{V}_{-j_2} + \frac{1}{2}\sum_{j=1}^{\infty}(\overline{\varepsilon}j - \widehat{V}_0)\widehat{V}_j\widehat{V}_{-j} + \frac{1}{6}\widehat{V}_0^3$$

Theorem [M. 2019b]: For $\overline{\varepsilon} \in \mathbb{R}$, $\hbar > 0$, the eigenvalues $E_\lambda(\overline{\varepsilon}, \hbar)$ found by [Stanley 1989] are *renormalized* **energies** of **quantized** traveling waves.

- The $\hbar$-*Bohr-Sommerfeld conditions* on $v^{\vec{I},\vec{x}}(x,t;\varepsilon_0)$ – namely, that actions $\varepsilon_0^2 I_j \in 2\pi\hbar\mathbb{Z}$ – are equivalent to the conditions that I_j are determined by a **partition** $\lambda = (0 \le \cdots \le \lambda_2 \le \lambda_1)$ according to

$$\boxed{I_j = \frac{2\pi\hbar}{\varepsilon_0^2}\Big(\lambda_j - \lambda_{j+1}\Big)}$$

- The eigenvalues $E_\lambda(\overline{\varepsilon}, \hbar)$ of quantum BO Hamiltonian $\widehat{H}(\overline{\varepsilon}, \hbar)$ on $\mathbb{T}$ are ***exactly* the classical energies of the $\hbar$-BS quasi-periodic orbits** $v^{\vec{I},\vec{x}}(x,t;\varepsilon_0)$ after the *renormalization* in Abanov-Wiegmann (2005):

$$\boxed{\overline{\varepsilon} = \varepsilon_0\Big(1 - \frac{\hbar}{\varepsilon_0^2}\Big)}$$

⋆ Compare: quantization of sine-Gordon soliton and breather spectrum ⋆

Summary

1. What is a "quantum system"?
   - Ingredients: Hilbert space F of states Ψ & self-adjoint operators $\widehat{H}$, $\widehat{O}$
   - Born rule: outcome of measurements are random variables $\widehat{O}|_\Psi$
   - Dynamics: deterministic in isolation, stochastic upon observation

2. The quantization of classical BO equation $\partial_t v + \partial_x v = -\frac{\varepsilon_0}{2} J[\partial_x^2 v]$ on $\mathbb{T}$
   - Ingredients: Hilbert space F_R and quantum Hamiltonian $\widehat{H}(\overline{\varepsilon}, \hbar)$
   - [Stanley 1989]: Eigenfunctions of $\widehat{H}(\overline{\varepsilon}, \hbar)$ are *Jack polynomials* indexed by **partitions** λ with eigenvalues $E_\lambda(\overline{\varepsilon}, \hbar)$ via *Young diagrams.*

3. Recent results in quantization of classical BO periodic traveling waves
   - Gérard-Kappeler 2019: identify $\varepsilon_0^2 I_j$ as classical actions in known formulas for integrable ensembles $v^{\vec{I},\vec{x}}$ of BO periodic traveling waves.
   - Theorem: [M. 2019b] (I) **partitions** λ from $\hbar$-Bohr-Sommerfeld quantization of actions $\varepsilon_0^2 I_j$ with wavenumber j: $\boxed{I_j = \frac{2\pi\hbar}{\varepsilon_0^2}(\lambda_j - \lambda_{j+1})}$

     (II) eigenvalues $E_\lambda(\overline{\varepsilon}, \hbar)$ in [Stanley 1989] are **exactly** classical energies of **quantized** $v^{\vec{I},\vec{x}}$ after *renormalization*: $\boxed{\overline{\varepsilon} = \varepsilon_0(1 - \hbar/\varepsilon_0^2)}$

Next: how can we see quantum BO on $\mathbb{T}$ is "1-loop exact"?

Expect: in some "renormalization scheme", derive order $\hbar$ correction

$$\overline{\varepsilon} \approx \varepsilon_0 \left(1 - \frac{\hbar}{\varepsilon_0^2}\right)$$

by considering weak quantum fluctuations in transverse directions.

Recall: For classical BO with coupling $\varepsilon_0 \in \mathbb{R}$, derived $\hbar$-BS conditions

$$\boxed{I_j = \frac{2\pi\hbar}{\varepsilon_0^2}(\lambda_j - \lambda_{j+1})}$$

on classical N-phase solutions $v^{\vec{I},\vec{x}}(x,t;\varepsilon_0)$. These are at "0-loop". Why?

- If $I_1, \ldots, I_N > 0$, $v^{\vec{I},\vec{x}}(x,t;\varepsilon_0)$ is classical integrable system of interacting periodic traveling waves with wavenumbers $j = 1, \ldots, N$.
- Generically, under no-resonance conditions on classical frequencies ω_j, $v^{\vec{I},\vec{x}}(x,t;\varepsilon_0)$ explores N-dimensional real torus in phase space.
- Above $\hbar$-BS conditions neglect the ∞-many transverse directions in phase space associated to wavenumbers $j = N+1, N+2, \ldots$.

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by considering weak quantum fluctuations in transverse directions.

Strategy: use known results for stability of periodic traveling wave!

1. Let $v^{(I_1,\chi_1)}(x,t;\varepsilon_0)$ be classical BO periodic traveling wave with wavenumber $N = 1$. This wave has only $N = 1$ bump on $\mathbb{T}$.
2. *Linearize* the classical BO flow around periodic orbit $v^{(I_1,\chi_1)}(x,t;\varepsilon_0)$. Result: linear classical Hamiltonian system of *non-interacting* small amplitude disturbances riding fixed ambient periodic traveling wave.
3. Linearity: small disturbances built from *sinusoidal waves*. Since $N = 1$, disturbances with wavenumbers $j \neq 1$ from transverse directions.
4. Ambrose-Wilkening (2008): found linearized spectrum of this flow.
5. Quantize this linear classical Hamiltonian system. How? Wavenumbers $j = 1, 2, 3, \ldots$ index *independent harmonic oscillators* in transverse directions whose frequencies are determined by linearized spectrum.
6. **Missing:** use a "renormalization scheme" to detect $-\hbar/\varepsilon_0^2$ at "1-loop".

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Expect: in some "renormalization scheme", derive order $\hbar$ correction

$$\overline{\varepsilon} \approx \varepsilon_0 \left(1 - \frac{\hbar}{\varepsilon_0^2} \right)$$

by considering weak quantum fluctuations in transverse directions.

Previous results: in [M.2017] and [M.2020], showed that

- Coherent state quantization of any classical initial data $\phi(x,0)$

$$\Upsilon_{\phi(x,0)}(V_1, V_2, \ldots ; \hbar) = \exp\left(\frac{1}{\hbar} \sum_{k=1}^{\infty} \frac{\overline{\phi_k} V_k}{k} \right).$$

- Born Rule: observe quantum BO in state $\Upsilon_{\phi(x,0)}$ with $\widehat{H}(\overline{\varepsilon}, \hbar)$, encounter random partitions defined by Jack polynomials.

New evidence: [M.2021] If one takes $\phi(x,0) = v^{(I_1,\chi_1)}(x,0;\varepsilon_0)$ to be initial data of classical BO periodic traveling wave with wavenumber $N=1$, then above construction using renormalized $\overline{\varepsilon}$ **corresponds precisely to the** $N=1$ degenerate series **of** mixed z-measures **of** Borodin-Olshanski (2005)!

- In "$N=1$ degenerate series", almost surely $\lambda_j = 0$ for all $j \neq 1$!
- Meaning: *If* $\varepsilon_0 \to \overline{\varepsilon}$, *no quantum fluctuations in transverse directions!!*

Thank You!

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