New results on the structure of Jack Littlewood-Richardson coefficients.

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(articles [M.-Moll] (out today!) + [M.] & [Alexandersson-M.] to appear)

Jack Symmetric Functions

(Integral) Jack symmetric functions $J_\lambda^{(\alpha)}$ give an orthogonal basis of the ring of symmetric functions (in infinitely many variables x_i), indexed by partitions λ and a deformation parameter $\alpha \in \mathbb{R}$. They are deformations of Schur functions s_λ $(\alpha = 1)$.

Expressed in terms of the power-sum symmetric functions $p_k := \sum_i x_i^k$.

$$\begin{aligned} J^{(\alpha)}_{\{1^3\}}(p) &= \quad p_1^3 - 3p_2p_1 + 2p_3 \\ J^{(\alpha)}_{\{1,2\}}(p) &= \quad p_1^3 + (\alpha-1)p_2p_1 - \alpha p_3 \\ J^{(\alpha)}_{\{3\}}(p) &= \quad p_1^3 + 3\alpha p_2p_1 + 2\alpha^2 p_3 \end{aligned}$$

Long standing problem of understanding products:

$$J_\lambda \cdot J_\nu = \sum_\gamma c^\gamma_{\lambda,\nu}(\alpha)\, J_\gamma. \tag{1}$$

The Littlewood-Richardson coefficients $c^\gamma_{\lambda,\nu}(\alpha)$ have a deep structure. E.g. In the Schur case they capture the intersection numbers of Schubert varieties in a Grassmanian.

Stanley's conjecture

In [Stanley '89], it was conjectured that the LR coefficients of Jack functions exhibited striking properties:

$$\langle J_\lambda \cdot J_\nu, J_\sigma \rangle = c^\sigma_{\lambda\nu}(\alpha) \times ||J_\sigma||^2 \in \mathbb{Z}_{\geq 0}[\alpha]$$

Example: $\langle J_{12} J_{12}, J_{123} \rangle = 6\alpha^4(2+\alpha)(1+2\alpha)(2+11\alpha+2\alpha^2)$

It's since been proven [Knop-Sahi '96] that these are integral polynomials, but positivity still open!

Stanley also provided a stronger version of this conjecture in the case where $c^\sigma_{\lambda\nu}(1) = 1$, here the LR coefficient should factorize as a product of linear factors (hook lengths):

$$\langle J_{2,3} \cdot J_{2^2}, J_{4,5} \rangle = 96\alpha^5(1+\alpha)^3(2+\alpha)^2(1+2\alpha)(1+3\alpha)(1+4\alpha)(1+5\alpha).$$

Strong version proved in several cases:
Pieri rule: $\langle J_{1^r} \cdot J_\mu, J_\nu \rangle$ [Stanley '89],
Rectangular case: $\langle J_\mu \cdot J_\nu, J_{m^n} \rangle$ [Cai-Jing '13] (also for $m^n - 1^r$)

New Approach: $1+1$ dimensional Integrable QFT

The Benjamin-Ono equation ['67,'75]:

$$v_t + v\, v_x = \bar{\varepsilon}\, \mathcal{H}[v_{xx}]$$

A 1+1 dimensional classical field theory describing deep water waves, for a periodic field $v = \sum_{n=-\infty}^{\infty} v_n(t) e^{inx} : S^1_x \times \mathbb{R}_t \to \mathbb{R}$. $\mathcal{H}$ is the Hilbert transform ($\pm i$ on $\pm$ Fourier modes), a *non-local* operator. $\bar{\varepsilon}$ is a dispersion parameter, and $\bar{\varepsilon} \to 0$ is the Hopf equation. The BO equation admits an infinite family of integrals of motion, which are given in terms of the following (semi-infinite) Lax matrix,

$$L(t) := \pi_{\geq 0}\left(v(x,t) - i\bar{\varepsilon}\partial_x\right) = \begin{pmatrix} 0 & v_1 & v_2 & \cdots \\ v_{-1} & \bar{\varepsilon} & v_1 & \cdots \\ v_{-2} & v_{-1} & 2\bar{\varepsilon} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Theorem [Nakamura, Bock-Kruskal '79]: $\exists$ a matrix M, s.t the Lax eqn $\partial_t L = [M, L]$ is equivalent to the BO eqn.

Classical Lax equation and Integrability

L is not functional class, so we can't consider the usual $\mathrm{tr}(L^n)$. Instead, consider the top-corner matrix element of powers of L

$$t_n := [L^n]_{00}$$

$$t_2 = \sum_{n>0} v_n v_{-n}, \qquad t_3 = H_{BO} = \sum_{n,m=0}^{\infty} v_n v_{n-m} v_{-m} + \overline{\varepsilon} \sum_{n=0}^{\infty} n v_n v_{-n}$$

Theorem [Nazarov-Skylanin '13]: $\{t_n\}$ are conserved quantities which Poisson commute w.r.t $\{,\}$.

Quantization

Nazarov-Skylanin remarkably prove this classical statement via its quantization! The classical field $v(x, t)$, with modes that satisfy

$$\{v_{-n}, v_m\} = in\delta_{m,n}$$

is quantized to the $\hat{\mathfrak{gl}}_1$ current $\hat{v}(z) = \sum_n V_n z^n$ with

$$[V_{-n}, V_m] = \hbar n\delta_{m,n}$$

The modes of this current act on the Fock module

$$\mathcal{F} = \mathbb{C}[V_1, V_2, \ldots]$$

with $V_{k>0}$ via multiplication and

$$V_{-k} = \hbar\, k \frac{\partial}{\partial V_k}$$

Quantum Lax Operator and Integrability

Definition: The Nazarov-Sklyanin Quantum BO Lax operator

$$\mathcal{L} := \begin{pmatrix} 0 & V_1 & V_2 & V_3 & \cdots \\ V_{-1} & \overline{\varepsilon} & V_1 & V_2 & \cdots \\ V_{-2} & V_{-1} & 2\overline{\varepsilon} & V_2 & \cdots \\ V_{-3} & V_{-2} & V_{-1} & 3\overline{\varepsilon} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{2}$$

acting on the extended Fock module $\mathcal{H} = \mathcal{F} \otimes \mathbb{C}[w]$.

$$\mathcal{L} \equiv \mathcal{L}(\hbar, \overline{\varepsilon}) = \pi_{\geq 0} \sum_k w^{-k} V_k + \hbar \sum_k k w^k \frac{\partial}{\partial V_k} + \overline{\varepsilon} w \frac{\partial}{\partial w}$$

Theorem [Nazarov-Sklyanin 2013]: The operators $\mathcal{T}_n = [\mathcal{L}^n]_{00}$ are a family of commuting quantum Hamiltonians acting on $\mathcal{F}$, that quantize the classical hamiltonians t_n.

$$\mathcal{T}_2 = \hbar \mathcal{N} = \sum_{n>0} V_n V_{-n}, \qquad \mathcal{T}_3 = H_{qBO} = \sum_{n,m=0}^{\infty} V_n V_{n-m} V_{-m} + \overline{\varepsilon} \sum_{n=0}^{\infty} n V_n V_{-n}$$

Spectrum of Quantum Hamiltonians

Let

$$\mathcal{T}(u) = \left[\tfrac{u}{u-\mathcal{L}}\right]_{00} = \sum_{n>0} u^{-n}\mathcal{T}_n$$

Theorem [Nazarov-Sklyanin '13, Macdonald for $\mathcal{T}_3$] The operator $\mathcal{T}(u) : \mathcal{F} \to \mathcal{F}$ decomposes $\mathcal{F}$ into eigenfunctions j_λ satisfying

$$\mathcal{T}(u) j_\lambda = T_\lambda(u) j_\lambda$$

- $j_\lambda \sim J_\lambda^{(\alpha)}$ are Jack symmetric functions with α related to $\bar{\varepsilon}, \hbar$ with a slightly different normalization: using $V_k = (-\varepsilon_2)p_k$, and

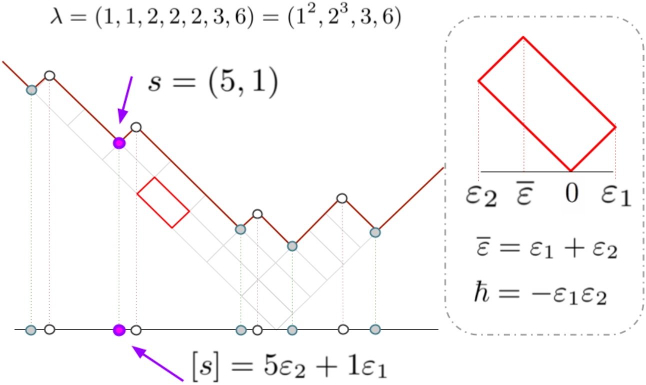
  $$j_\lambda(V|\varepsilon_1, \varepsilon_2) := (-\varepsilon_2)^{|\lambda|} \cdot J_\lambda^{(\alpha=-\varepsilon_1/\varepsilon_2)}(p)$$

  which gives e.g.

  $$j_{1,2}(V|\varepsilon_1, \varepsilon_2) = V_1^3 + (\varepsilon_1 + \varepsilon_2)V_1 V_2 + \varepsilon_1\varepsilon_2 V_3 \in \mathcal{F}_3$$

Partitions, profiles and minima

For a partition λ of an integer n, we draw the following profile:



λ made up of boxes s, indexed by the bottom corner grid point. Skewness of the profile given by two anisotropy factors, $\varepsilon_2 < 0 < \varepsilon_1 \in \mathbb{R}$.
Set of minima of the partition profile: $\min \lambda$. The *content* of the box at s is $[s] := s.(\varepsilon_2, \varepsilon_1)$. The deformation parameter is $\alpha = -\varepsilon_1/\varepsilon_2$.

Spectrum of Quantum Hamiltonians II

Theorem [Nazarov-Sklyanin 2013] The family $\{\mathcal{T}_n\}$ (w/ $\overline{\varepsilon} = \varepsilon_1 + \varepsilon_2$, $\hbar = -\varepsilon_1\varepsilon_2$) is diagonalized on Jack symmetric functions,

$$\mathcal{T}(u) j_\lambda = T_\lambda(u) \cdot j_\lambda$$

where

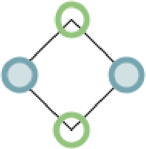
$$\begin{aligned} T_\lambda(u) &= \prod_{s\in\lambda} T_1(u-[s]), \qquad T_1(u) := \frac{(u-[0,0])(u-[1,1])}{(u-[1,0])(u-[0,1])} \\ &= u\,\frac{\prod_{t\in\max\lambda}(u-[t])}{\prod_{s\in\min\lambda}(u-[s])}, \\ &= \sum_{s\in\min\lambda} \frac{\hat{\tau}^s_\lambda}{u-[s]}, \qquad \hat{\tau}^s_\lambda := \operatorname*{Res}_{u=[s]} T_\lambda(u) \end{aligned}$$

Note: j_λ is not an eigenvector of $\mathcal{L}$.
Residues are Kerov's well known *transition measures* :

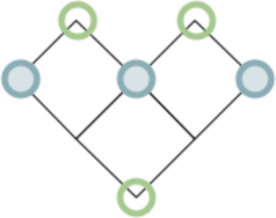
$$\tau^s_\lambda := [s]^{-1}\hat{\tau}^s_\lambda = \operatorname*{Res}_{u=[s]} u^{-1} T_\lambda(u), \qquad \sum_s \tau^s_\lambda = 1$$

Spectrum of Quantum Hamiltonians II

$T_1(u) = \frac{(u-[0,0])(u-[1,1])}{(u-[1,0])(u-[0,1])}$



$T_{\{1,2\}}(u) = T_1(u)T_1(u-[1,0])T_1(u-[0,1]) = \frac{(u-[0,0])(u-[2,1])(u-[1,2])}{(u-[2,0])(u-[1,1])(u-[0,2])}$



Spectral Theorem

Our new work begins with a deeper study of this quantum Lax operator. Note that $\mathcal{L}$ commutes with the total grading operator $\mathcal{N} + w\partial_w$. Let $\mathcal{H}_n = (\mathcal{F} \otimes \mathbb{C}[w])_n$ be the graded components.

Spectral Theorem [M.-Moll '22]: The eigenvectors of $\mathcal{L}$ on $\mathcal{H}_n$ are given by orthogonal states

$$\psi_\lambda^s \quad \text{for } \lambda \in Part(n), s \in \min \lambda$$

with

$$\mathcal{L}\,\psi_\lambda^s = [s]\psi_\lambda^s$$

Furthermore, these can be normalized such that (π_0 : projection onto w^0)

$$\pi_0 \psi_\lambda^s = j_\lambda$$

e.g.

$$\psi_3^{(0,1)} = j_3 + (\varepsilon_2 V_1^2 + \varepsilon_1 \varepsilon_2 V_2)w + 2\varepsilon_1\varepsilon_2 V_1 w^2 + 2\varepsilon_1^2 \varepsilon_2 w^3$$

$$\mathcal{L}\psi_3^{(0,1)} = \varepsilon_2 \psi_3^{(0,1)}$$

Conjecture (in-progress) $\psi_\lambda^s \in \mathbb{Z}[V, w, \varepsilon_1, \varepsilon_2]$.

The cyclic subspace of $\mathcal{H}_n$ generated by j_λ under the action of $\mathcal{L}$, is given by

$$\mathcal{Z}_\lambda := Z(j_\lambda, \mathcal{L}) = \mathrm{Span}_{s \in \min \lambda} \psi_\lambda^s$$

$$j_\lambda = \sum_{s \in \min \lambda} \tau_\lambda^s \psi_\lambda^s \ \in \mathcal{Z}_\lambda, \qquad \tau_\lambda^s = [s]^{-1} \hat{\tau}_\lambda^s$$

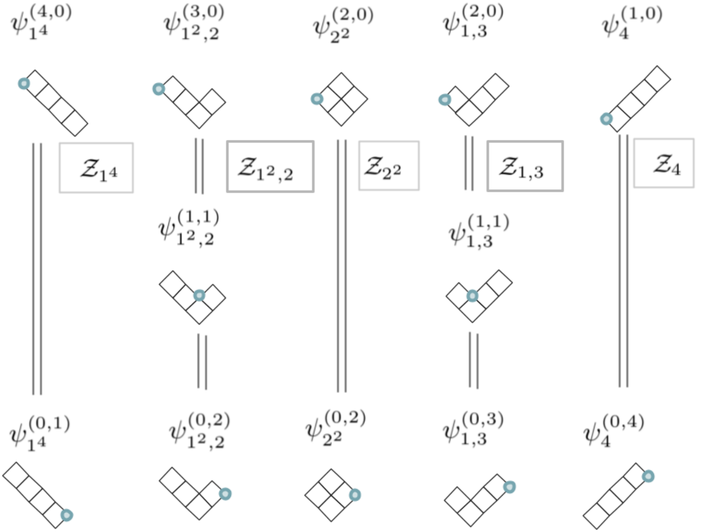
Coefficients of Jacks in Lax eigenfunction basis are transition measures!

These cyclic subspaces give us a decomposition of the graded Hilbert space

$$\mathcal{H}_n = \bigoplus_{\lambda \in Part(n)} \mathcal{Z}_\lambda$$

This is the first of three such decompositions.

Spectral Theorem: $\mathcal{Z}$ decomposition of $\mathcal{H}_4$



Three Decompositions

The second decomposition is given by the splitting into eigenspaces

$$\mathcal{Y}^s = \{\psi^s_\lambda\}_{\lambda : s \in \min \lambda}$$

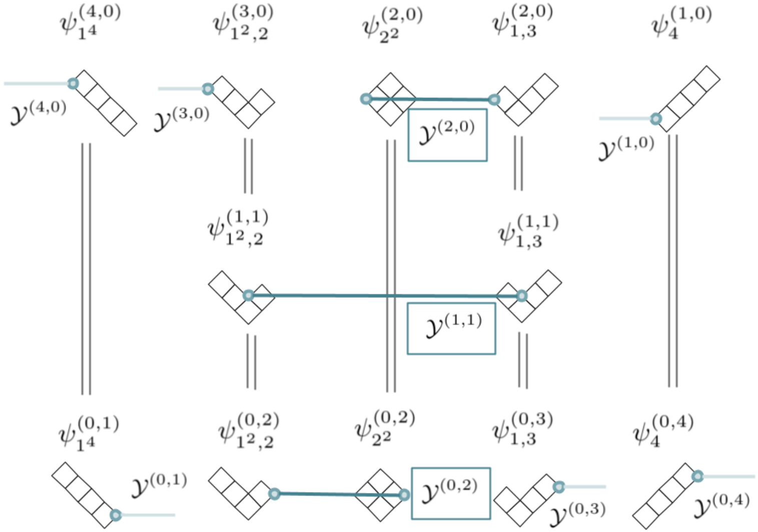
Example:

$$\mathcal{Y}_4^{(2,0)} = \mathbb{C}\psi_{2^2}^{(2,0)} \oplus \mathbb{C}\psi_{1,3}^{(2,0)}$$

We have

$$\mathcal{H}_n = \bigoplus_{s \in \mathbb{N}^2} \mathcal{Y}_n^s$$

Spectral Theorem: $\mathcal{Y}$ decomposition



Action of w

The third decomposition involves the action of w on Lax eigenvectors

Theorem [M. '22]

$$w \cdot \psi_\lambda^s = \sum_{t \in \min \lambda + s} \frac{\tau_{\lambda+s}^t}{[s - t + (1,1)]} \psi_{\lambda+s}^t$$

(recall $[s] = s.(\varepsilon_2, \varepsilon_1)$)

Notice that $w \cdot \psi_\lambda^s \in \mathcal{Z}_{\lambda+s}$.

Corollary:

$$\psi_\lambda^s = \sum_{\mu \subseteq \lambda} a_{\lambda,\mu;s}\, j_\mu\, w^{|\lambda|-|\mu|}$$

e.g.

$$\psi_3^{(0,1)} = j_3 + (\varepsilon_2) j_2 w + (2\varepsilon_1\varepsilon_2) j_1 w^2 + (2\varepsilon_1^2\varepsilon_2) j_\emptyset w^3$$

So for $\gamma \in Part(n+1)$, define

$$\mathcal{X}_\gamma = \bigoplus_{(\lambda,s):\lambda+s=\gamma} \mathbb{C}\psi_\lambda^s \subset \mathcal{H}_n \tag{3}$$

As a consequence of our $w\psi$ theorem, we find $w \cdot \mathcal{X}_\gamma \subset \mathcal{Z}_\gamma$. In fact

$$\mathcal{Z}_\gamma = \mathbb{C}j_\gamma \oplus w \cdot \mathcal{X}_\gamma$$

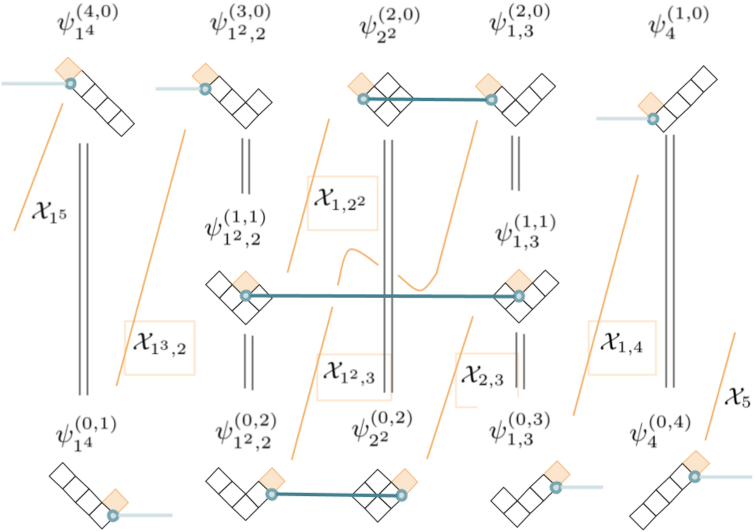
Third decomposition:

$$\mathcal{H}_n = \bigoplus_{|\gamma|=n+1} \mathcal{X}_\gamma$$

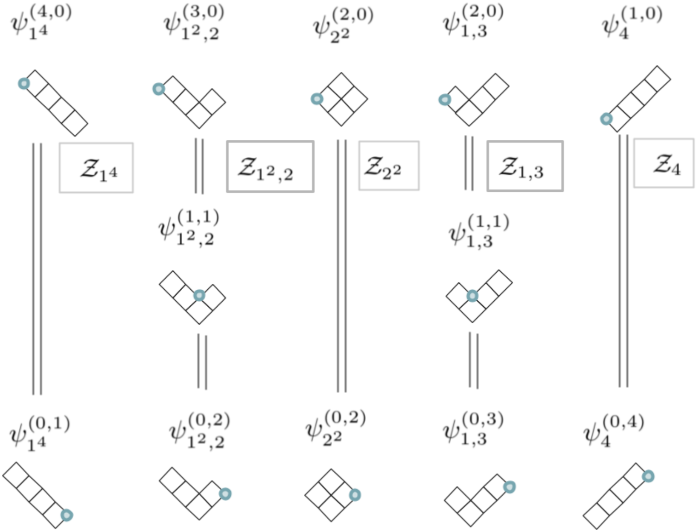
Note that any two of these subspaces of $\mathcal{H}_n$ have at most one-dimensional intersection, in which case

$$\mathcal{Z}_\lambda \cap \mathcal{Y}^s \cap \mathcal{X}_{\lambda+s} = \mathbb{C}\psi_\lambda^s$$

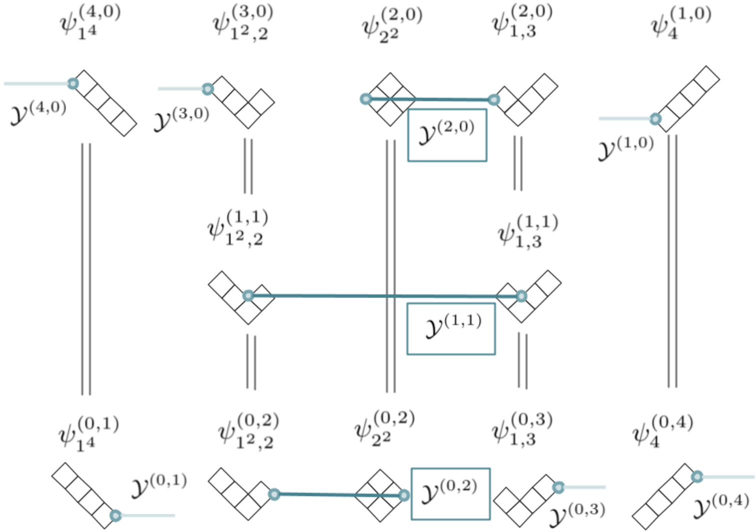
Spectral Theorem: $\mathcal{X}$ decomposition



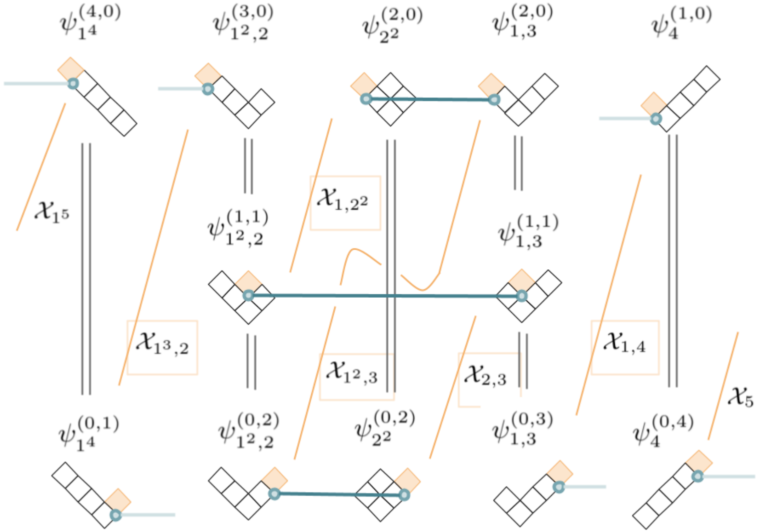
Three Decompositions



Three Decompositions



Three Decompositions



Products

(more on these decompositions in a moment)

Returning now to the original question of products of Jack functions

$$j_\lambda \cdot j_\nu = \sum_\gamma c^\gamma_{\lambda,\nu} j_\gamma. \tag{4}$$

In [M.] it is suggested to approach this problem by considering the products of Lax eigenfunctions

$$\psi^s_\lambda \cdot \psi^t_\nu = \sum_{\gamma,u} c^{s,t;\gamma}_{\lambda,\nu;u} \psi^u_\gamma. \tag{5}$$

This recovers the Jack LR coefficients under π_0.

Refined Pieri rule for Lax eigenfunctions

E.g. using our result for $w\psi$, we find the simplest $\psi\psi$ product:

$$\psi^v_{\{1\}} \cdot \psi^s_\lambda = \sum_{u \in \min \lambda + s} \frac{[-v][s-u-v+(1,1)]}{[s-u][s-u+(1,1)]} \tau^u_{\lambda+s} \psi^u_{\lambda+s} + \sum_{t \in \min \lambda, t \neq s} \tau^t_\lambda \psi^s_{\lambda+t}$$

π_0 of this recovers the simplest 'Pieri rule' for Jack LR coefficients [Stanley '89, Kerov '93]

$$j_{\{1\}} \cdot j_\lambda = \sum_{t \in \min \lambda} \tau^t_\lambda \, j_{\lambda+t}$$

Big theme: Relationship between Jack LR coeffs and residues of rational functions.

$$c^{\lambda+t}_{1,\lambda} = \tau^t_\lambda$$

But what do we gain by enlarging this algebra? Is this any simpler?

We saw that when we project the eigenfunctions ψ_λ^s onto w^0 (π_0, i.e. set $w \to 0$), we recover the Jack functions j_λ. We can also consider the map π_* which takes the coefficient of the top power of w on each graded component (i.e. set all $V_k \to 0, w \to 1$).
We can show

$$\pi_* \psi_\lambda^s = \prod_{t \in \lambda + s, t \neq (0,0)} [t] \; = [s] \cdot ([V_{|\lambda|}] j_\lambda)$$

Recall the NS result (which is about π_0)

$$\pi_0 \frac{u}{u - \mathcal{L}} j_\lambda = T_\lambda(u) \cdot j_\lambda$$

On the other hand, we can show a complementary result about π_*:

$$\pi_* \frac{1}{u - \mathcal{L}} j_\lambda = (T_\lambda(u) - 1) \left([V_{|\lambda|}] j_\lambda \right)$$

$$\frac{1}{u - \mathcal{L}} j_\lambda = u^{-1} T_\lambda(u) \cdot j_\lambda + \ldots + (T_\lambda(u) - 1) \left([V_{|\lambda|}] j_\lambda \right) w^{|\lambda|}$$

We define the 'y-trace' as a linear functional on $\zeta \in \mathcal{H} = \mathcal{F} \otimes \mathbb{C}[w]$.

$$\mathrm{y}_u(\zeta) := \pi_* \frac{1}{u - \mathcal{L}} \zeta \in \mathbb{C}_{\varepsilon_1, \varepsilon_2}(u),$$

We change normalisation for our basis vectors to make expressions simpler under π_*:

$$\hat{\psi}^s_\lambda = \psi^s_\lambda / \pi_* \psi^s_\lambda, \qquad \hat{j}_\lambda = j_\lambda / ([V_{|\lambda|}] j_\lambda)$$

$$\hat{\tau}^s_\lambda := \operatorname*{Res}_{u=[s]} T_\lambda(u) = [s] \tau^s_\lambda$$

With these new normalisations, we have

$$\hat{j}_\lambda = \sum_{s \in \min \lambda} \hat{\tau}^s_\lambda \hat{\psi}^s_\lambda$$

$$\mathrm{y}_u(\hat{j}_\lambda) = T_\lambda(u) - 1, \qquad \mathrm{y}_u(\hat{\psi}^s_\lambda) = \frac{1}{u - [s]}$$

FUNCTIONALS

Although the general form of $\psi\psi$ products are quite complicated, the y-traces are surprisingly simple

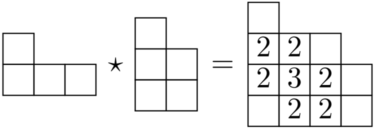
Theorem (M. '22)

$$\mathrm{y}_u\left(\hat{\psi}^s_\lambda \cdot \hat{\psi}^t_\nu\right) = \frac{T_{\lambda\star\nu}(u)}{u - [s+t]}$$

where

$$T_{\lambda\star\nu}(u) := \prod_{a\in\lambda, b\in\nu} T_1(u - [a+b])$$

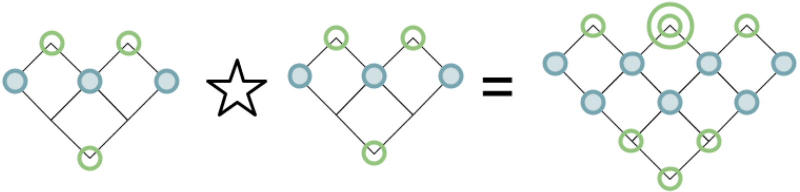
$$T_1(u) := \frac{(u-[0,0])(u-[1,1])}{(u-[1,0])(u-[0,1])}$$



Example:

$$T_{\{1,2\}\star\{1,2\}}(u) =$$

$$\frac{(u-[3,1])(u-[2,2])^2(u-[1,3])(u-[1,0])(u-[0,0])(u-[0,1])}{(u-[3,0])(u-[2,0])(u-[2,1])(u-[1,1])(u-[1,2])(u-[0,2])(u-[0,3])}$$



$$\mathrm{y}_u(\hat{\psi}^{(1,1)}_{1,2}\hat{\psi}^{(2,0)}_{1,2}) = \frac{T_{\{1,2\}\star\{1,2\}}(u)}{(u-[3,1])}$$

Unique Minima

Note there is a kernel: if both λ_1 and λ_2 have s as a minima, then

$$\mathrm{y}_u\left(\hat{\psi}^s_{\lambda_1} - \hat{\psi}^s_{\lambda_2}\right) = 0$$

so the y-functional formula only constrains the $\psi\psi$ product.

However, it does determine some coefficients: when a minima s appears only in one particular partition of size n (in which case we call s *unique*).

The minima $(\delta, 0)$ on $\gamma = \{1^\delta\}$ and $(\delta - 1, 0)$ on $\gamma = \{1^{\delta-1}, 2\}$ are unique.

If $\delta = mn - 1$, then $(m-1, n-1)$ is unique on $\gamma = \{m-1, m^{n-1}\}$. We call such γ *sub-rectangular* partitions (rectangular minus one box).

If v is a unique minima, on a partition γ, then

$$\hat{\psi}^s_\lambda \cdot \hat{\psi}^t_\nu = \cdots + \operatorname*{Res}_{u=[v]}\left(\mathrm{y}_u\left(\hat{\psi}^s_\lambda \hat{\psi}^t_\nu\right)\right)\hat{\psi}^v_\gamma + \cdots$$

Return to Jacks

Recall $\hat{j}_\lambda = \sum_{s\in\min\lambda} \hat{\tau}^s_\lambda \hat{\psi}^s_\lambda$, so

$$\begin{aligned} \mathrm{y}_u\left(\hat{j}_\lambda \cdot \hat{j}_\nu\right) &= \sum_{s\in\min\lambda, t\in\min\nu} \hat{\tau}^s_\lambda \hat{\tau}^t_\nu \mathrm{y}_u\left(\hat{\psi}^s_\lambda \hat{\psi}^t_\nu\right) \\ &= T_{\lambda\star\nu}(u) \sum_{s\in\min\lambda, t\in\min\nu} \frac{\hat{\tau}^s_\lambda \hat{\tau}^t_\nu}{u-[s+t]} \end{aligned}$$

So we apply y_u to the Jack product expansion (now normalized by $\hat{j}_\lambda = j_\lambda/([V_{|\lambda|}]j_\lambda))$

$$\hat{j}_\lambda \cdot \hat{j}_\nu = \sum_\gamma \hat{c}^\gamma_{\lambda,\nu} \hat{j}_\gamma. \tag{6}$$

we find an equality of rational functions

$$\mathrm{y}_u\left(\hat{j}_\lambda \cdot \hat{j}_\nu\right) = \sum_\gamma \hat{c}^\gamma_{\lambda,\nu} \mathrm{y}_u\left(\hat{j}_\gamma\right) \tag{7}$$

$$T_{\lambda\star\nu}(u) \sum_{s\in\min\lambda, t\in\min\nu} \frac{\hat{\tau}^s_\lambda \hat{\tau}^t_\nu}{u-[s+t]} = \sum_\gamma \hat{c}^\gamma_{\lambda,\nu}(T_\gamma(u)-1) \tag{8}$$

Note: This determines all $\hat{c}^\gamma_{\lambda,\nu}$ for $|\gamma| \le 7$. But this formula is cumbersome.

More functionals

We can do better. To say more, we'll need more functionals.

We already saw the y_u trace, which measures projection onto $\mathcal{Y}^r$ subspaces

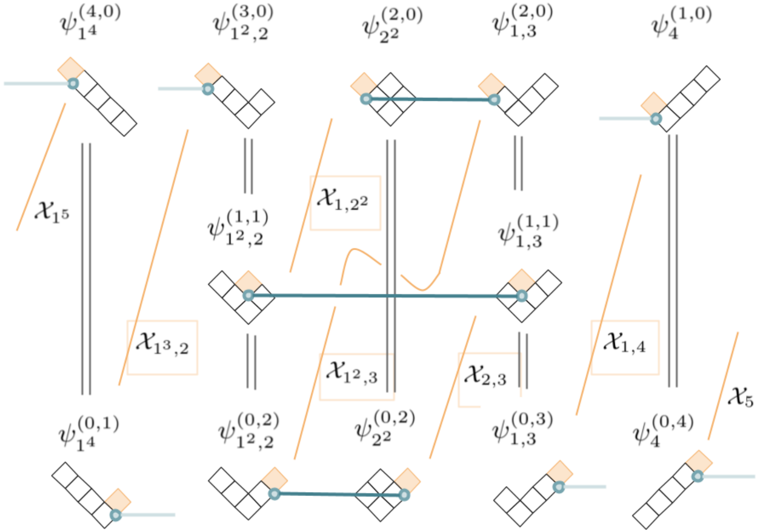
$$\mathrm{y}^r(\zeta) := \operatorname*{Res}_{u=[r]} \mathrm{y}_u(\zeta) = \pi_* P_{\mathcal{Y}^r} \zeta$$

We can also consider two other traces made out of projections on the other two decompositions of $\mathcal{H}_n$:

$$\mathrm{z}_\lambda(\zeta) = \pi_* P_{\mathcal{Z}_\lambda} \zeta, \qquad \mathrm{x}_\gamma(\zeta) = \pi_* P_{\mathcal{X}_\gamma} \zeta, \tag{9}$$

All of these are maps from $\mathcal{H} \to R := \mathbb{C}(\varepsilon_1, \varepsilon_2)$.

More functionals



More functionals

We combine all these traces into

$$\mathrm{Tr}_n := \oplus_\gamma \mathrm{x}_\gamma \oplus_r \mathrm{y}^r \oplus_\lambda z_\lambda$$

which sits in an exact sequence

$$\ker \mathrm{Tr}_n \to \mathcal{H}_n \xrightarrow{\mathrm{Tr}_n} R^{p(n+1)} \oplus R^{q(n)} \oplus R^{p(n)} \to \operatorname{coker} \mathrm{Tr}_n$$

$(R := \mathbb{C}(\varepsilon_1, \varepsilon_2))$

Easy to see that $\ker \mathrm{Tr}_n$ is given by expressions of the form

$$\Gamma_\eta^{a,b,c} := \hat{\psi}^c_{\eta+a} - \hat{\psi}^c_{\eta+b} + \hat{\psi}^b_{\eta+c} - \hat{\psi}^b_{\eta+a} + \hat{\psi}^a_{\eta+b} - \hat{\psi}^a_{\eta+c}$$

where $a, b, c \in \min \eta$ with $|\eta| = n - 1$.

More on the cokernel shortly.

Special elements

The formula

$$\mathrm{y}_u\left(\hat{\psi}^s_\lambda \hat{\psi}^t_\nu\right) = \frac{T_{\lambda\star\nu}(u)}{u-[s+t]},$$

can be written alternatively as

$$\mathrm{y}_u\left(\beta^{s,t}_{\lambda,\nu}\right) = T_{\lambda\star\nu}(u) - 1$$

where

$$\beta^{s,t}_{\lambda,\nu} = (\mathcal{L} - [s+t])(\hat{\psi}^s_\lambda \hat{\psi}^t_\nu) \in \mathcal{H}_{|\lambda|+|\nu|}.$$

These turn out to be extremely interesting elements of the algebra:

Theorem (M.)

$\mathrm{Tr}(\beta^{s,t}_{\lambda,\nu})$ *is given by*

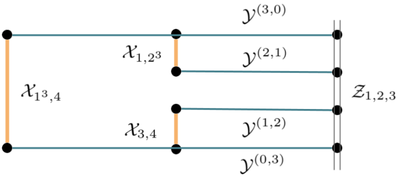
$$\mathrm{x}_\gamma(\beta^{s,t}_{\lambda,\nu}) = 0, \quad \mathrm{y}^r(\beta^{s,t}_{\lambda,\nu}) = \hat{\tau}^r_{\lambda\star\nu}, \quad \mathrm{z}_\sigma(\beta^{s,t}_{\lambda,\nu}) = -\hat{c}^\sigma_{\lambda,\nu}.$$

This implies that elements in $\operatorname{coker}\mathrm{Tr}_n$, i.e. relations between $\mathrm{x}_\gamma, \mathrm{y}^r, \mathrm{z}_\sigma$, give relations between the Jack LR's $\hat{c}^\sigma_{\lambda,\nu}$ and the residues $\hat{\tau}^r_{\lambda\star\nu}$!

The cokernel: Examples

The following is a relation in coker Tr_6:

$$\mathrm{z}_{123} - \mathrm{y}^{(3,0)} - \mathrm{y}^{(2,1)} - \mathrm{y}^{(1,2)} - \mathrm{y}^{(0,3)} + \mathrm{x}_{12^3} + \mathrm{x}_{34} + \mathrm{x}_{1^34} = 0$$



Applying this to $\mathrm{Tr}(\beta^{*,*}_{12,12})$ gives,

$$-\hat{c}^{123}_{12,12} = \hat{\tau}^{(3,0)}_{12\star12} + \hat{\tau}^{(2,1)}_{12\star12} + \hat{\tau}^{(1,2)}_{12\star12} + \hat{\tau}^{(0,3)}_{12\star12}$$

$$T_{12\star12}(u) = \frac{(u-[3,1])(u-[2,2])^2(u-[1,3])(u-[1,0])(u-[0,0])(u-[0,1])}{(u-[3,0])(u-[2,0])(u-[2,1])(u-[1,1])(u-[1,2])(u-[0,2])(u-[0,3])}$$

and we can easily compute:

$$\langle j_{12}j_{12}, j_{123}\rangle = 6\alpha^4(2+\alpha)(1+2\alpha)(2+11\alpha+2\alpha^2)$$

The cokernel

Theorem [M.]: coker Tr_n is given by the span of the residues of the rational function

$$C_n(u) := \mathrm{y}_u - \sum_{\gamma \vdash n+1} \mathrm{x}_\gamma \left(\sum_{t \in \gamma} \frac{1}{u - [t]} \right) + \sum_{\lambda \vdash n} \mathrm{z}_\lambda \left(\sum_{s \in \lambda} \frac{1}{u - [s]} \right). \qquad (10)$$

[Note: easy to check this holds for $\mathrm{Tr}(\hat{\psi}_\lambda^s)$, just need to show that this exhausts the cokernel]

Main Result

Applying this full cokernel relation to the trace of the β element, i.e. $C_n(u)(\mathrm{Tr}_n(\beta^{s,t}_{\lambda,\nu}))$, yields the following striking relation

$$\sum_{\gamma \vdash n : \mu,\nu \subseteq \gamma} \hat{c}^{\gamma}_{\mu\nu} \left(\sum_{s \in \gamma/(\mu \cup \nu)} \frac{1}{u - [s]} \right) = T_{\mu \star \nu}(u) - 1. \tag{11}$$

This is a two-parameter generalization of Kerov's result ($c^{\mu+t}_{\mu,1} = \tau^t_\mu$), i.e.

$$\sum_{t \in \min \mu} \hat{c}^{\mu+t}_{\mu,1} \left(\frac{1}{u - [t]} \right) = T_\mu(u) - 1. \tag{12}$$

A peculiar map

Another way of phrasing this previous result is in terms of the following map on symmetric functions $\Delta : \Lambda \to \mathbb{Q}_\varepsilon(u)$, defined on the basis of (integral) Jacks as

$$\Delta(J_\lambda) := \varpi_\lambda \sum_{s\in\lambda} \frac{1}{u-[s]}, \qquad \text{where} \ \ \varpi_\lambda := [p_{|\lambda|}]J_\lambda = \prod_{s\in\lambda^\times} [s]. \tag{13}$$

This map satisfies

$$\Delta(J_\mu \cdot J_\nu) = \varpi_\mu \varpi_\nu \left(T_{\mu\star\nu}(u) - 1\right). \tag{14}$$

Note: Not clear how to prove this simple statement without using $\mathcal{L}$.

Explicit LR formulae: Rectangular Unions

With this identity we can find explicit expressions for certain Jack LR coefficients.

Let μ be arbitrary, and m^n be a rectangular partition. Let $\mu \cup m^n$ be the union of the two partitions (as collections of boxes), and $\overline{\sigma}$ be the reverse of $(\mu \cup m^n/\mu)$, (i.e. rotated by 180 degrees).

Corollary:

$$\hat{c}^{\mu \cup m^n}_{\mu,\overline{\sigma}} = \mathrm{res}_{u=[n-1,m-1]} T_{\mu \star \overline{\sigma}}(u)$$

Stanley's Conjecture

Conjecture [Stanley 89]: if $c^{\gamma}_{\mu\nu}(1) = 1$, then there exists choices of U/L hook lengths

$$\langle J_\mu J_\nu, J_\gamma \rangle = \prod_{a \in \mu} h^{U/L}(a) \prod_{b \in \nu} h^{U/L}(b) \prod_{c \in \gamma} h^{U/L}(c)$$

Even though we have an explicit form of the LR coefficient for the rectangular union case, its not obvious that such an expression in terms of hook lengths exists.

Theorem [Alexandersson-M '22] The strong Stanley conjecture holds in the rectangular union case.

Agrees with rectangular case from [Cai-Jing '13] where $\mu \subset m^n$.

Summary + Future Directions

Results:

[M.-Moll] Spectral theorem for Nazarov-Skylanin Lax $\mathcal{L}$.

[M.] Structural theorems about algebra of eigenfunctions (y-trace, properties β elements), cokernel relation.

[Alexandersson-M.] Strong version of Stanley conjecture for rectangular unions

To do:

- Show polynomiality of ψ_λ^s (in-progress) + more algebra structure.
- Corresponding statements for Schur functions, LR coeffs.
- Further progress on Stanley's conjectures.
- Extend to Macdonald polynomials. (q, t) analogue of $\mathcal{L}$ was described in [Nazarov-Sklyanin 2019].
- Understanding of ψ_λ^s in relation to equivariant geometry of Hilbert scheme $\mathrm{Hilb}_n(\mathbb{C}^2)$.

Thank you!